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Free longitudinal vibration analysis of multi-step non-uniform bars based on piecewise analytical solutions

Q.S. Li *

Department of Building and Construction, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong

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Abstract

Using appropriate transformations, the differential equations of free longitudinal vibrations of bars with variably distributed mass and stiffness are reduced to Bessel's equations or ordinary differential equations with constant coefficients by selecting suitable expressions, such as power functions and exponential functions, for the distributions of stiffness and mass. Exact analytical solutions to determine the longitudinal natural frequencies and mode shapes for a one step non-uniform bar are derived and used to obtain the frequency equation of a multi-step non-uniform bar with several boundary conditions. This approach which combines the transfer matrix method and closed-form solutions of one step non-uniform bars leads to a single frequency equation for any number of steps. Numerical example shows that the computed values of the longitudinal fundamental natural frequency and mode shape of a tall building by the proposed method are close to the field measured data. It is also demonstrated through the numerical example that the selected expressions are suitable for describing the distributions of stiffness and mass of typical tall buildings.

Keywords: Longitudinal vibration; Non-uniform bars; Tall buildings; Natural frequency; Mode shape

1. Introduction

The longitudinal vibration of non-uniform bars is a subject of considerable scientific and practical interest that has been studied extensively. For example, the analysis of longitudinal vibration of non-uniform bars is important in the design of foundation. Meek and Wolf [1–3] presented an extensive set of solutions for 'cone models' (in essence, tapered rods) for foundation design. Wang (Ref. [4] pp. 320–25) reported that the magnitude of the vertical component of ground motion is often about one-third of the horizontal component, and the vertical component of ground motions has a significant effect on earthquake induced responses of high-rise structures. Thus, it is necessary to determine the natural frequencies and mode shapes in vertical direction for high-rise structures at design stage for certain cases. When analysing free vibrations of high-rise structures, it is possible to regard such structures as a cantilever bar with varying cross-section [4–6]. The solution for free longitudinal vibration of uniform structural members is well known. However, in general, it is not possible or, at least, very difficult to get the exact analytical solutions of differential equations for free vibrations of bars with variably distributed mass and stiffness. These exact bar solutions are available only for certain bar shapes and boundary conditions. Conway et al. ([7] obtained an exact solution for a conical beam in terms of Bessel functions. Wang [4] derived the closed-form solutions for the free longitudinal vibration of a bar with variably distributed stiffness and mass that were described by exponential functions. Bapat [8] obtained exact solutions for the free longitudinal vibration of exponential and catenoidal rods. Lau [9] and Abrate [10] derived closed-form solutions for the free longitudinal vibration of rods whose cross-section varies as $A(x)=A_0(x/L)^2$ and $A(x)=A_0[1+a(x/L)]^2$, respectively. Kumar et al. [11] found exact solutions for the free longitudinal vibration of non-uniform rods whose cross-section varies as $A(x)=(a+bx)^n$ and $A(x)=A_0\sin^2(ax+b)$. The natural frequencies of such rods for various end conditions were calculated, and their dependence on taper was discussed.

* Tel.: +852-2784-4677; fax: +852-2788-7612.
E-mail address: bcqsli@cityu.edu.hk (Q.S. Li).

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Nomenclature

y	displacement in the longitudinal direction
$p(x,t)$	intensity of axial force
$X(x)$	mode shape function
ω	circular natural frequency
$\bar{m}_x$	mass per unit length at section x
K_x	axial stiffness at section x
H	height of the structure
α	longitudinal stiffness at x=0
a	mass per unit length at x=0
β,γ,c	constants in Eq. (3)

In order to find the closed-form solutions for the longitudinal free vibration of non-uniform rods or bars, it is usually assumed that the mass of a rod or a bar is proportional to its stiffness (e.g. Wang [4], Li et al. [6], Kumar et al. [11]). This calculation model is reasonable for a part of high-rise structures. However, this assumption is not valid for tall buildings and many high-rise structures. This is due to the fact that the mass of floors is 80% or even more of the total mass of a tall building, and the variation of mass at different floors is not significant. So, the mass distribution with height is almost constant for many cases, suggesting that the value of mass of a tall building is not necessarily proportional to its stiffness. This is confirmed by a series of shaking tests on buildings of various types in which the mass and stiffness of individual buildings have been measured and reported [5,12,13]. In this paper, exact analytical solutions for free longitudinal vibrations of one-step bars with variably distributed stiffness and mass, in which the value of mass is not necessarily proportional to its stiffness, are proposed. The derived analytical solutions are used to obtain the frequency equation of a multi-step non-uniform bar with several boundary conditions. This approach which combines the transfer matrix method and closed-form solutions of one step non-uniform bars leads to a single frequency equation for any number of steps.

Apart from the several analytical methods for analysing limited classes of non-uniform rods or bars, many approximate methods have been developed. These include the Ritz method, the finite strip method (FSM) and the finite element method (FEM). In general, the Ritz method can provide accurate solutions, however, it depends on the choice of global admissible functions. Liew and his co-workers [14–18] have developed efficient three-dimensional Ritz algorithms for the free vibration analysis of elastic solid cylinders. Their method, developed based on a global three-dimensional elasticity energy principle with polynomial-based displacement shape functions, is capable of extracting all possible modes of vibration for elastic solid cylinders. Their work provided useful benchmarking reference for research development in simplified beam theories because three-dimensional analysis is an important base for exact comparison studies. The FEM and FSE have been developed and widely applied to vibration analysis of various non-uniform structural members over the years. Compared with FEM, the main advantage of FSE is its efficiency, in particular for structural members with regular geometry.

The objective of this paper is to present exact analytical solutions for the free longitudinal vibrations of bars with variably distributed stiffness and mass. In the absence of exact solutions, this problem can be solved using approximated methods (e.g. the Ritz method) or numerical methods (e.g. FEM). However, the present exact solutions could provide adequate insight into the physics of the problem and can be easily implemented. The availability of the exact solutions will help in examining the accuracy of the approximate or numerical solutions. Therefore, it is always desirable to obtain the exact solutions to such problems.

2. Free longitudinal vibrations of one-step bars

The governing differential equation for longitudinal (or axial) vibration of a one-step bar with variable cross-section (Fig. 1) can be established as follows

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial y}{\partial x}\right)=\bar{m}_x\frac{\partial^2 y}{\partial t^2}+p(x,t) \quad (1)$$

in which y, $p(x,t)$, K_x and $\bar{m}_x$ are the displacement in the longitudinal direction, the intensity of axial force, axial stiffness and mass per unit length, respectively, at section x.

If $p(x,t)$=0, then, Eq. (1) becomes the equation of free longitudinal vibration.

Setting

$$y(x,t)=X(x)\sin(\omega t+\gamma_0) \quad (2)$$

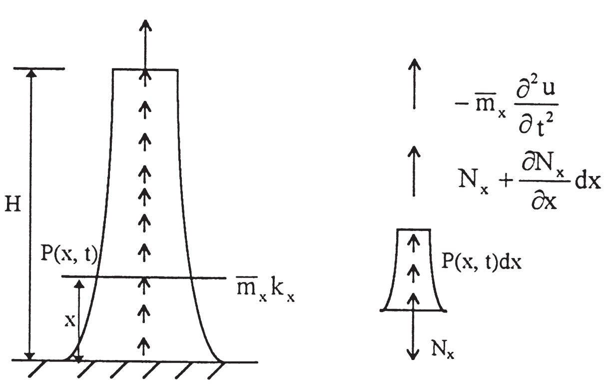

Fig. 1. A cantilever bar with variable cross-section.

where ω is the circular natural frequency and γ_0 is the initial phase.

Then, the equation of mode shape function, $X(x)$, is given by

$$K_x\frac{d^2X}{dx^2}+\frac{dK_x}{dx}\frac{dX}{dx}+\bar{m}_x\omega^2X=0 \tag{3}$$

It is difficult to find the analytical solutions of Eq. (3) for general cases, because the structural parameters in the equation vary with the co-ordinate x. It is obvious that the analytical solutions are dependent on the distributions of mass and stiffness. Thus, the analytical solution of Eq. (3) may be obtained by means of reasonable selections for mass and stiffness distributions. As suggested by Wang [4], Tuma and Cheng [19] and Li et al. [20–22], the functions that can be used to approximate the variation of mass and stiffness are algebraic polynomials, exponential functions, trigonometric series, or their combinations. In this paper, two important cases are considered and discussed as follows.

Case A: Expressions of mass and axial stiffness are power functions as follows

$$K_x=\alpha\left(1+\beta\frac{x}{H}\right)^{\gamma} \tag{4}$$

$$\bar{m}_x=a\left(1+\beta\frac{x}{H}\right)^{c} \tag{5}$$

in which a, c, α, β and γ are constants which can be determined by use of the real values of the axial stiffness and mass intensity at x=0, H/2 and H as follows

$$\left.\begin{array}{c}\alpha=K_0\quad a=\bar{m}_0\\ \beta=\left(\frac{K_H}{K_0}\right)^{\frac{1}{\gamma}}-1\quad b=\left(\frac{\bar{m}_H}{\bar{m}_0}\right)^{\frac{1}{c}}-1\\ \gamma=\frac{\ln K_{\frac{H}{2}}-\ln K_0}{\ln\left(1+\frac{\beta}{2}\right)}\quad c=\frac{\ln\bar{m}_{\frac{H}{2}}-\ln\bar{m}_0}{\ln\left(1+\frac{\beta}{2}\right)}\end{array}\right\} \tag{6}$$

where m_0, K_0, $m_{\frac{H}{2}}$, $K_{\frac{H}{2}}$, m_H, K_H are the mass intensity and the axial stiffness, respectively, at x=0, H/2, and H. H is the height of the structure considered (Fig. 1).

It can be seen from Eq. (6) that α is the longitudinal stiffness at x=0, β and γ represent the taper of the non-uniform bar.

Substituting Eqs. (4) and (5) into Eq. (3) gives

$$\frac{d^2X}{dx^2}+\frac{\gamma\beta}{H\left(1+\beta\frac{x}{H}\right)}\frac{dX}{dx}+\frac{a\omega^2}{\alpha}\left(1+\beta\frac{x}{H}\right)^{c-\gamma}X=0 \tag{7}$$

Setting

$$\left.\begin{array}{c}X=\left(1+\beta\frac{x}{H}\right)^{\frac{1-\gamma}{2}}\psi\\ \xi=\frac{2n}{c-\gamma+2}\left(1+\beta\frac{x}{H}\right)^{\frac{c-\gamma+2}{2}}\\ n^2=\frac{a\omega^2H^2}{\alpha\beta^2}\end{array}\right\} \tag{8}$$

Substituting Eq. (8) into Eq. (7) one yields

$$\frac{d^2\psi}{d\xi^2}+\frac{1}{\xi}\frac{d\psi}{d\xi}+\left(1-\frac{\nu^2}{\xi^2}\right)\psi=0 \tag{9}$$

where $\nu=\frac{1-\gamma}{c-\gamma+2}$

Eq. (9) is a Bessel's equation of the νth order. The general solution of vibration mode shape function and the eigenvalue equation are as follows

1. For a non-integer ν

$$X(x)=\left(1+\beta\frac{x}{H}\right)^{\frac{1-\gamma}{2}}[c_1J_\nu(\xi)+c_2J_{-\nu}(\xi)] \tag{10}$$

where $J_\nu(\xi)$ is the Bessel function of the first kind of order ν.

The eigenvalue equation for this case is

$$J_\nu(\lambda)J_{-(\nu-1)}(\lambda\theta)=-J_{-\nu}(\lambda)J_{\nu-1}(\lambda\theta)\text{ for a cantilever bar} \tag{11}$$

or

$$J_\nu(\lambda)J_{-\nu}(\lambda\theta)=J_{-\nu}(\lambda)J_\nu(\lambda\theta)\text{ for a fixed}-\text{fixed bar} \tag{12}$$

where

$$\left.\begin{array}{c}\lambda=\frac{2n}{c-\gamma+2}\\ \theta=(1+\beta)^{\frac{c-\gamma+2}{2}}\end{array}\right\} \tag{13}$$

Solving the eigenvalue equation obtains the jth eigenvalue, λ_j (j=1,2,...), and substituting λ_j into Eqs. (8) and (13) one yields the jth circular natural frequency, ω_j, and the jth mode shape as follows

$$\omega_j=\frac{(c-\gamma+2)|\beta|\lambda_j}{2H}\sqrt{\frac{\alpha}{a}} \tag{14}$$

$$X_j(x)=\left(1+\beta\frac{x}{H}\right)^{\frac{1-\gamma}{2}}\left\{J_\nu\left[\lambda_j\left(1+\beta\frac{x}{H}\right)^{\frac{c-\gamma+2}{2}}\right]-\frac{J_\nu(\lambda_j)}{J_{-\nu}(\lambda_j)}J_{-\nu}\left[\lambda_j\left(1+\beta\frac{x}{H}\right)^{\frac{c-\gamma+2}{2}}\right]\right\} \tag{15}$$

2. For an integer ν

$$X(x)=\left(1+\beta\frac{x}{H}\right)^{\frac{1-\gamma}{2}}[c_1J_\nu(\xi)+c_2Y_\nu(\xi)] \quad (16)$$

where $Y_\nu(\xi)$ is the Bessel function of the second kind of order ν.

The eigenvalue equation for this case is

$$J_\nu(\lambda)Y_{\nu-1}(\lambda\theta)=Y_\nu(\lambda)J_{\nu-1}(\lambda\theta) \text{ for a cantilever bar} \quad (17)$$

or

$$J_\nu(\lambda)Y_\nu(\lambda\theta)=Y_\nu(\lambda)J_\nu(\lambda\theta) \text{ for a fixed}-\text{fixed bar} \quad (18)$$

The jth circular natural frequency, ω_j, can be determined by Eq. (14), and the jth mode shape can be expressed as follows

$$X_j(x)=\left(1+\beta\frac{x}{H}\right)^{\frac{1-\gamma}{2}}\left\{J_\nu\left[\lambda_j\left(1+\beta\frac{x}{H}\right)^{\frac{c-\gamma+2}{2}}\right] -\frac{J_\nu(\lambda_j)}{J_{-\nu}(\lambda_j)}Y_\nu\left[\lambda_j\left(1+\beta\frac{x}{H}\right)^{\frac{c-\gamma+2}{2}}\right]\right\} \quad (19)$$

The following special cases can be found from the general solution presented above.

1. If γ=1, then, ν=0, setting ν=0 in Eqs. (16)–(19) obtains the eigenvalue equations and mode shape functions for this case.
2. If γ=c+2, then Eq. (7) is reduced to an Euler's equation as follows

$$\left(1+\beta\frac{x}{H}\right)^2\frac{d^2X}{dx^2}+\gamma\beta\left(1+\beta\frac{x}{H}\right)\frac{dX}{dx}+\frac{a\omega^2}{\alpha}X=0 \quad (20)$$

The general solution of Eq. (20) can be written as

$$X(x)=\left(1+\beta\frac{x}{H}\right)^{\frac{1-\gamma}{2}}\left\{c_1\cos\left[\sqrt{D}\ln\left(1+\beta\frac{x}{H}\right)\right] +c_2\sin\left[\sqrt{D}\ln\left(1+\beta\frac{x}{H}\right)\right]\right\} \quad (21)$$

For a cantilever bar, the general solution of Eq. (20) is as follows

$$X(x)=\left(1+\beta\frac{x}{H}\right)^{\frac{1-\gamma}{2}}\sin\left[\sqrt{D}\ln\left(1+\beta\frac{x}{H}\right)\right] \quad (22)$$

The eigenvalue equation is

$$2\sqrt{D}\text{ctn}\left[\sqrt{D}\ln(1+\beta)\right]=\gamma-1 \quad (23)$$

in which

$$D=n^2-\frac{(1-\gamma)^2}{4}>0 \quad (24)$$

Because the case corresponding to $D<0$ is meaningless, it is not considered here.

3. If γ=c, then, ν=1−γ/2, the general solution becomes that of a bar in which the mass of it is proportional to its longitudinal stiffness. In general, solid bars and some high-rise structures belong to this case.
4. When $\gamma\neq0$, c=0, this case represents a bar with variably distributed stiffness and uniformly distributed mass. The corresponding solution can be found from the general solution. For this case, ν=(1−γ)/(2−γ). Some tall buildings can be considered as this case.
5. When γ=0, c$\neq$0, the general solution becomes that of a bar with uniformly distributed stiffness and variably distributed mass. For this case, ν=1/(2+c).
6. When β=−1, the general solution becomes the solution of a wedged bar with variably distributed stiffness and variably distributed mass (Fig. 2).
7. When γ=0 and c=0, the general solution represents that of a uniform bar. The general solution for this case can be written as [4]

$$X(x)=c_1\cos\eta\frac{x}{H}+c_2\sin\eta\frac{x}{H} \quad (25)$$

in which

$$\eta^2=\frac{aH^2\omega^2}{\alpha}$$

The jth circular natural frequency, ω_j, and the jth mode shape are as follows

$$\left.\begin{aligned}\omega_j&=\frac{(2j-1)\pi}{2H}\sqrt{\frac{\alpha}{a}}\\ X_j(x)&=\cos\frac{(2j-1)\pi x}{2H}\end{aligned}\right\}\text{For a cantilever bar} \quad (26)$$

or

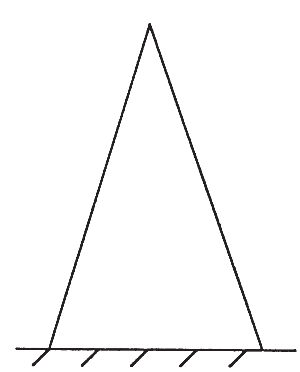

Fig. 2. A cuneiform bar.

$$\left.\begin{aligned} \omega_j &= \frac{j\pi}{H}\sqrt{\frac{\alpha}{a}} \\ X_j(x) &= \sin\frac{j\pi x}{H} \end{aligned}\right\} \text{For a fixed}-\text{fixed bar} \quad (27)$$

Case B: Expressions of mass and axial stiffness are exponential functions as follows

$$K_x = \alpha e^{-\beta\frac{x}{H}} \quad (28)$$

$$m_x = a e^{-b\frac{x}{H}} \quad (29)$$

The parameters α,β,a,b can be determined by

$$\alpha = K_0,\ \beta = \ln(K_0) - \ln(K_H),\ a = m_0,\ b = \ln(m_0) - \ln(m_H) \quad (30)$$

Substituting Eqs. (28) and (29) into Eq. (3) gives

$$\frac{d^2X}{dx^2} - \frac{\beta}{H}\frac{dX}{dx} + \frac{a}{\alpha}\omega^2 e^{\frac{(\beta-b)x}{H}} X = 0 \quad (31)$$

Setting

$$\left.\begin{aligned} X &= \xi^{\nu} Z \\ \xi &= e^{\frac{(\beta-b)x}{2H}} \\ \nu &= \frac{\beta}{\beta-b} \\ \lambda^2 &= \frac{4a\omega^2H^2}{\alpha(\beta-b)^2} \end{aligned}\right\} \quad (32)$$

Substituting Eq. (32) into Eq. (31) leads to

$$\frac{d^2Z}{d\xi^2} + \frac{1}{\xi}\frac{dZ}{d\xi} + \left(\lambda^2 - \frac{\nu^2}{\xi^2}\right) Z = 0 \quad (33)$$

Eq. (33) is a Bessel's equation of the νth order.

1. For a non-integer ν

$$X(x) = e^{\frac{\beta x}{2H}}[c_1 J_\nu(\lambda e^{\frac{Ax}{H}}) + c_2 J_{-\nu}(\lambda e^{\frac{Ax}{H}})] \quad (34)$$

The eigenvalue equation is

$$J_\nu(\lambda) Y_{-(\nu-1)}(\lambda A) = -J_{-\nu}(\lambda) J_{\nu-1}(\lambda A) \text{ for a cantilever bar} \quad (35)$$

or

$$J_\nu(\lambda) J_{-\nu}(\lambda A) = J_{-\nu}(\lambda) J_\nu(\lambda A) \text{ for a fixed}-\text{fixed bar} \quad (36)$$

in which

$$A = e^{\frac{\beta-b}{2}} \quad (37)$$

Solving the eigenvalue equation one obtains the jth eigenvalue, λ_j (j=1,2,...), and substituting λ_j into Eq. (32) gives the jth circular natural frequency, ω_j, and the jth mode shape as follows

$$\omega_j = \frac{|\beta-b|\lambda_j}{2H}\sqrt{\frac{\alpha}{a}} \quad (38)$$

$$X_j(x) = e^{\frac{\beta x}{2H}}\left[J_\nu(\lambda_j e^{\frac{Ax}{H}}) - \frac{J_\nu(\lambda_j)}{J_{-\nu}(\lambda_j)} J_{-\nu}(\lambda_j e^{\frac{Ax}{H}})\right] \quad (39)$$

2. For an integer ν

$$X(x) = e^{\frac{\beta x}{2H}}[c_1 J_\nu(\lambda e^{\frac{Ax}{H}}) + c_2 Y_\nu(\lambda e^{\frac{Ax}{H}})] \quad (40)$$

The eigenvalue equation is

$$J_\nu(\lambda) Y_{\nu-1}(\lambda A) = Y_\nu(\lambda) J_{\nu-1}(\lambda A) \text{ for a cantilever bar} \quad (41)$$

or

$$J_\nu(\lambda) Y_\nu(\lambda A) = Y_\nu(\lambda) J_\nu(\lambda A) \text{ for a fixed}-\text{fixed bar} \quad (42)$$

Solving the eigenvalue equation one obtains the jth eigenvalue, λ_j (j=1,2,...), and substituting λ_j into Eq. (38) gives the jth circular natural frequency, ω_j. The jth mode shape for a cantilever bar and a fixed-fixed bar can be written as

$$X_j(x) = e^{\frac{\beta x}{2H}}\left[J_\nu(\lambda_j e^{\frac{Ax}{H}}) - \frac{J_\nu(\lambda_j)}{Y_\nu(\lambda_j)} Y_\nu(\lambda_j e^{\frac{Ax}{H}})\right] \quad (43)$$

The following special cases can be found from the general solution presented above.

1. When $\beta \neq 0$, b=0, it represents a bar with variably distributed stiffness and uniformly distributed mass. In this case ν=1.
2. When β=0, b≠0, it represents a bar with variably distributed mass and uniformly distributed stiffness. In this case ν=0.
3. When β=b, the general solution becomes that of a bar in which the mass of it is proportional to its longitudinal stiffness. For this case, Eq. (31) is reduced to a differential equation with constant coefficients as

$$\frac{d^2X}{dx^2} - \frac{\beta}{H}\frac{dX}{dx} + \mu^2 X = 0 \quad (44)$$

in which

$$\mu^2 = \frac{a\omega^2}{\alpha} \quad (45)$$

It is obvious that if

$$\frac{\beta^2}{H^2} - 4\mu^2 \geq 0 \quad (46)$$

then, only zero solution exists.When

$$\frac{\beta^2}{H^2}-4\mu^2<0 \quad (47)$$

The general solution of $X(x)$ is given by

$$X(x)=e^{\frac{\beta x}{2H}}\left(A_1\cos\frac{cx}{H}+A_2\sin\frac{cx}{H}\right) \quad (48)$$

where

$$c^2=\frac{H^2}{4}\left(4\mu^2-\frac{\beta^2}{H^2}\right) \quad (49)$$

The boundary conditions of a cantilever bar can be written as

$$X(x)=0 \text{ at } x=0 \quad (50a)$$

$$\frac{dX(x)}{dx}=0 \text{ at } x=H \quad (50b)$$

Substituting Eq. (50a) into Eq. (48) one yields A_1=0, then using Eqs. (50)b and (48) one obtains the eigenvalue equation for a cantilever bar as follows

$$\tan c=-\frac{2}{\beta}c \quad (51)$$

Solving the above equation one obtains a set of c_j (j=1,2,…), then substituting c_j into Eq. (49) yields the jth circular natural frequency as follows

$$\left.\begin{aligned}\omega_j&=\frac{1}{H}\sqrt{\frac{\alpha}{a}\left[c_j^2+\left(\frac{\beta}{2}\right)^2\right]}\\ \omega_j&\approx\frac{c_j}{H}\sqrt{\frac{\alpha}{a}}(j\geq 2)\end{aligned}\right\}\text{for a cantilever bar} \quad (52)$$

The boundary conditions of a fixed-fixed bar are given by

$$X(x)=0, \text{ at } x=0 \text{ and } x=H \quad (53)$$

Using Eqs. (53) and (48) one obtains the eigenvalue equation and circular natural frequencies of a fixed-fixed bar are as follows

$$\left.\begin{aligned}&\sin c=0,\ c_j=j\pi\\ &\omega_j=\frac{1}{H}\sqrt{\frac{\alpha}{a}\left[(j\pi)^2+\left(\frac{\beta}{2}\right)^2\right]}\\ &\omega_j\approx\frac{j\pi}{H}\sqrt{\frac{\alpha}{a}}(j\geq 2)\end{aligned}\right\} \quad (54)$$

The jth mode shape for a cantilever bar and a fixed-fixed bar can be written as

$$X_j(x)=e^{\frac{\beta x}{2H}}\sin\frac{c_j x}{H} \quad (55)$$

4. When β=b=0, this case represents a uniform bar.

3. Free longitudinal vibrations of multi-step bars

Although the general solutions derived above for one-step bars with variable cross-section can be used to determine natural frequencies and mode shapes of many structures, there are two problems to be solved. First, some structures consist of several steps (see Fig. 3). Second, the distributions of stiffness and mass of some structures may not obey the assumed expressions given in the above two cases. Such structures can be treated as multi-step bars. If the steps are divided appropriately, the distributions of stiffness and mass per unit length in each step may match accurately or approximately one of the expressions described in the last section. The analytical solution of a one-step bar with variable cross-section can be used to derive the general solution and the eigenvalue equation of a multi-step bar using the transfer matrix method. One of the advantages of the present method is that the total number of steps required could be much less than that normally used in the conventional finite element methods.

A multi-step bar is shown in Fig. 3. It is assumed that each step bar has variably distributed stiffness and mass. The equation of mode shape of the ith step bar is as follows

$$K_{ix}\frac{d^2X_i}{dx^2}+\frac{dK_{ix}}{dx}\frac{dX_i}{dx}+\bar{m}_{ix}\omega^2X_i=0\ (i=1,2,\ldots,q) \quad (56)$$

The general solution of the mode shape function of the displacement, $X_i(x)$, and that of the axial force, $N_i(x)$, of the ith step bar can be expressed as

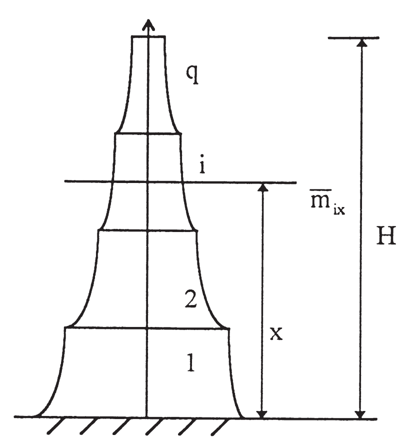

Fig. 3. A multi-step bar.

$$X_i(x)=C_{i1}S_{i1}(x)+C_{i2}S_{i2}(x)\ (i=1,2,\ldots,q) \tag{57}$$

$$N_i(x)=C_{i1}K_{ix}S'_{i1}(x)+C_{i2}K_{ix}S'_{i2}(x) \tag{58}$$

where i denotes the ith step bar and q is the number of steps of the bar divided (Fig. 3), $S_{i1}(x)$ and $S_{i2}(x)$ are special solutions of the mode shape of the ith step bar, and the prime denotes differentiation with respect to x.

If the stiffness and mass of the ith step bar are described by the power functions, Eqs. (4) and (5), then $S_{i1}(x)$ and $S_{i2}(x)$ can be found from Eq. (10) or Eq. (16). If the stiffness and mass of the ith step bar are described by the exponential functions, Eqs. (28) and (29), then $S_{i1}(x)$ and $S_{i2}(x)$ are given in Eq. (34) or Eq. (40). If the ith step bar is a uniform one, then, $S_{i1}(x)$ and $S_{i2}(x)$ can be found from Eq. (25).

The transfer matrix method is introduced herein to establish the mode shape equation and the eigenvalue equation of a multi-step bar (Fig. 3).

The mode shape functions of displacement $X_i(x)$ and the axial force $N_i(x)$, can be expressed as a matrix equation

$$\begin{bmatrix} X_i(x) \\ N_i(x) \end{bmatrix}=\begin{bmatrix} S_{i1}(x) & S_{i2}(x) \\ K_{ix}S'_{i1}(x) & K_{ix}S'_{i2}(x) \end{bmatrix}\begin{bmatrix} C_{i1} \\ C_{i2} \end{bmatrix} \tag{59}$$

According to Eq. (59) and considering the two ends of the ith step bar (Fig. 4), we have

$$\begin{bmatrix} X_{i0} \\ N_{i0} \end{bmatrix}=[S(x_{i0})]\begin{bmatrix} C_{i1} \\ C_{i2} \end{bmatrix} \tag{60}$$

$$\begin{bmatrix} X_{i1} \\ N_{i1} \end{bmatrix}=[S(x_{i1})]\begin{bmatrix} C_{i1} \\ C_{i2} \end{bmatrix} \tag{61}$$

From Eq. (60), one yields

$$\begin{bmatrix} C_{i1} \\ C_{i2} \end{bmatrix}=[S(x_{i0})]^{-1}\begin{bmatrix} X_{i0} \\ N_{i0} \end{bmatrix} \tag{62}$$

Substituting Eq. (62) into Eq. (61) leads to

$$\begin{bmatrix} X_{i1} \\ N_{i1} \end{bmatrix}=[T_i]\begin{bmatrix} X_{i0} \\ N_{i0} \end{bmatrix} \tag{63}$$

in which

$$\left.\begin{aligned} &[T_i]=[S(x_{i1})][S(x_{i0})]^{-1} \\ &[S(x_{i0})]=\begin{bmatrix} S_{i1}(x_{i0}) & S_{i2}(x_{i0}) \\ K_{i0}S'_{i1}(x_{i0}) & K_{i0}S'_{i2}(x_{i0}) \end{bmatrix} \\ &[S(x_{i1})]=\begin{bmatrix} (S_{i1}(x_{i1}) & S_{i2}(x_{i1}) \\ K_{i1}S'_{i1}(x_{i1}) & K_{i1}S'_{i2}(x_{i1}) \end{bmatrix} \\ &X_{i0}=X_i(x_{i0})\ X_{i1}=X_i(x_{i1}),\ K_{i0}=K_{ix}(x_{i0}),\ K_{i1}=K_{ix}(x_{i1}), \\ &N_{i0}=N_i(x_{i0}),\ N_{i1}=N_i(x_{i1}) \end{aligned}\right\} \tag{64}$$

$[T_i]$ is called the transfer matrix because it transfers the parameters at the end 0 to those at the end 1 in the ith step bar.

The equation for the top step (i=q, Fig. 3) can be established by use of Eq. (63) repeatedly as follows

$$\begin{bmatrix} X_{q1} \\ N_{q1} \end{bmatrix}=[T]\begin{bmatrix} X_{10} \\ N_{10} \end{bmatrix} \tag{65}$$

in which

$$[T]=[T_q][T_{q-1}]\ldots[T_1] \tag{66}$$

$[T]$ is a matrix which can be expressed as

$$[T]=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{67}$$

If there is a lumped mass (Fig. 5, m_i, attached to the ith step bar, then, the transfer matrix $[T_i]$ should be replaced by $[T_{mi}]$,

$$[T_{mi}]=\begin{bmatrix} 1 & 0 \\ -m_i\omega^2 & 1 \end{bmatrix}[T_i] \tag{68}$$

According to the boundary conditions

$$\left.\begin{aligned} &x=0,\ \ X_{10}=0 \\ &x=H\ \ N_{q1}=0 \end{aligned}\right\} \tag{69}$$

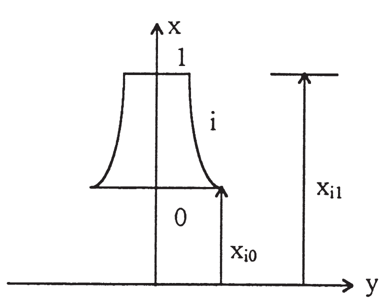

Fig. 4. Definition of the parameters at the two end of the ith step.

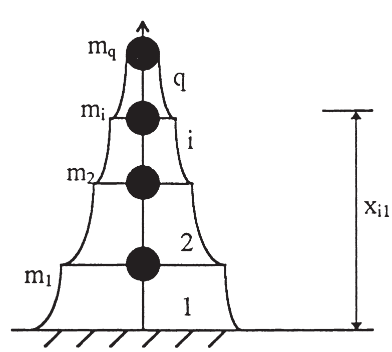

Fig. 5. A multi-step bar with lumped masses.

The eigenvalue equation for a multi-step cantilever bar is obtained as follows

$$T_{22}=0 \tag{70}$$

Using the boundary conditions

$$\left.\begin{matrix} x=0, & X_{10}=0 \\ x=H & x_{q1}=0 \end{matrix}\right\} \tag{71}$$

obtains the eigenvalue equation for a multi-step fixed-fixed bar as

$$T_{12}=0 \tag{72}$$

The mode shape function of a multi-step bar can be determined by use of Eq. (60) and the general solutions of each step bar after the natural frequencies of the bar have been found.

For a one-step bar (Fig. 6), the eigenvalue equation can be found by setting q=1, i.e. the transfer matrix, [T], should be replaced by

$$[T]=\begin{bmatrix}1 & 0\\ -\omega^2 m & 1\end{bmatrix}[T_1] \tag{73}$$

When the mass and stiffness are described by the power functions, the eigenvalue equation for a cantilever bar (Fig. 6) is

For a non-integer ν:

$$J_{-\nu}(\lambda)\left[n\theta BJ_{\nu-1}(\lambda\theta)+\frac{n^2\alpha^2\beta^2 m}{\mathrm{a}H^2}J_\nu(\lambda\theta)\right]= \tag{74}$$

$$-J_\nu(\lambda)\left[n\theta BJ_{-(\nu-1)}(\lambda\theta)-\frac{n^2\alpha^2\beta^2 m}{\mathrm{a}H^2}J_{-\nu}(\lambda\theta)\right]$$

For an integer ν:

$$Y_\nu(\lambda)\left[n\theta BJ_{\nu-1}(\lambda\theta)+\frac{n^2\alpha^2\beta^2 m}{\mathrm{a}H^2}J_\nu(\lambda\theta)\right] \tag{75}$$

$$=J_\nu(\lambda)\left[n\theta BY_{\nu-1}(\lambda\theta)-\frac{n^2\alpha^2\beta^2 m}{\mathrm{a}H^2}Y_\nu(\lambda\theta)\right]$$

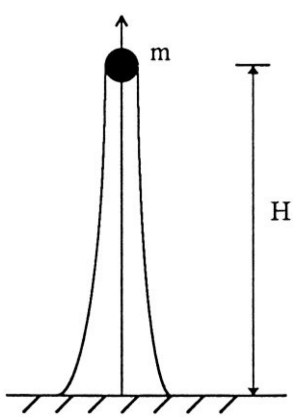

Fig. 6. A cantilever bar with a lumped mass.

where m is a mass attached on the top of the cantilever bar (Fig. 6) and $B=(1+\beta)^{\frac{1+\gamma}{2}}$

If γ=c+2, then, the eigenvalue equation is

$$2\sqrt{D}\mathrm{ctn}[\sqrt{D}\ln(1+\beta)]=\gamma-1-\frac{2\alpha\beta(1+\beta)n^2 m}{\mathrm{a}H} \tag{76}$$

When the mass and stiffness are described by the exponential functions, the eigenvalue equation for a cantilever bar (Fig. 6) is

For a non-integer ν:

$$J_{-\nu}(\lambda)\left[\frac{(\beta-\mathrm{b})\lambda m}{2\mathrm{a}H}J_\nu(\lambda A)+A\mathrm{e}^{-\beta}J_{\nu-1}(\lambda A)\right] \tag{77}$$

$$=J_\nu(\lambda)\left[\frac{(\beta-\mathrm{b})\lambda m}{2\mathrm{a}H}J_{-\nu}(\lambda A)-A\mathrm{e}^{-\beta}J_{-(\nu-1)}(\lambda A)\right]$$

For an integer ν:

$$Y_\nu(\lambda)\left[\frac{(\beta-\mathrm{b})\lambda m}{2\mathrm{a}H}J_\nu(\lambda A)+A\mathrm{e}^{-\beta}J_{\nu-1}(\lambda A)\right] \tag{78}$$

$$=J_\nu(\lambda)\left[\frac{(\beta-b)\lambda m}{2\mathrm{a}H}Y_\nu(\lambda A)+A\mathrm{e}^{-\beta}Y_{\nu-1}(\lambda A)\right]$$

After the jth natural frequency, ω_j (j=1,2,…), is determined by solving the eigenvalue equation, the jth mode shape can be found by substituting ω_j into Eq. (63) and using the boundary conditions. For example, setting $X_{10}=0$, $N_{10}=1$, i=1, and using Eq. (63) obtain X_{11} and N_{11}, then, X_{i1} and N_{i1} (i=1,2,3,…q) can be found by using Eq. (63), repeatedly, for a multi-step cantilever bar.

4. Numerical example

The main structure of Guangzhou Hotel Building is a R.C. shear-wall structure with 24 stories. There is a 3-storey appendage that is built on the top of the main structure. Based on the full-scale measurement of free vibration of this building [5], this building can be treated as a stepped cantilever bar (Fig. 7) in free vibration analysis. The major parameters of this building are listed in Table 1. The transfer matrix method developed in this paper is adopted herein to determine the longitudinal fundamental natural frequency and mode shape. Because the ith step bar considered herein is treated as an uniform one, $S_{i1}(x)$ and $S_{i2}(x)$ are given by Eq. (25), i.e.

$$S_{i1}(x)=\cos\eta_i\frac{x}{H}\;S_{i2}(x)=\sin\eta_i\frac{x}{H} \tag{79}$$

Table 1
Structural parameters and longitudinal fundamental mode shape of the building[a,b]

i	x_i(m)		m_i(kg/m)	K_i ($\times 10^9$ kN)	$X_i(x_i)$		
1	x_{10}	0			0	(0)	[0]
	x_{11}	5.35	35 010.2	133.14	0.1018	(0.100)	[0.1022]
2	x_{20}	5.35			0.1018	(0.100)	[0.1022]
	x_{21}	15.25	41 438.8	123.68			
3	x_{30}	15.25			0.2686	(0.257)	[0.2687]
	x_{31}	21.25	40 877.6	110.31	0.2686	(0.257)	[0.2687]
4	x_{40}	21.25			0.4267	(0.417)	[0.4269]
	x_{41}	33.85	39 337.7	97.71	0.4267	(0.417)	[0.4269]
					0.5686	(0.560)	[0.5687]
5	x_{50}	33.85					
	x_{51}	43.15	38 117.1	91.20	0.5686	(0.560)	[0.5687]
					0.7147	(0.710)	[0.7147]
6	x_{60}	43.15					
	x_{61}	52.45	36 295.3	82.32	0.7147	(0.710)	[0.7147]
					0.8474	(0.837)	[0.8475]
7	x_{70}	52.45					
	x_{71}	61.75	34 663.3	74.42	0.8474	(0.837)	[0.8475]
					0.9375	(0.929)	[0.9375]
8	x_{80}	61.75					
	x_{81}	76.00	35 536.1	69.27	0.9375	(0.929)	[0.9375]
					1.0000	(1.000)	[1.0000]

[a] The values in parentheses are the field measured data.
[b] The values in square brackets are the values calculated based on the model of a one-step cantilever bar with continuously varying stiffness and mass.

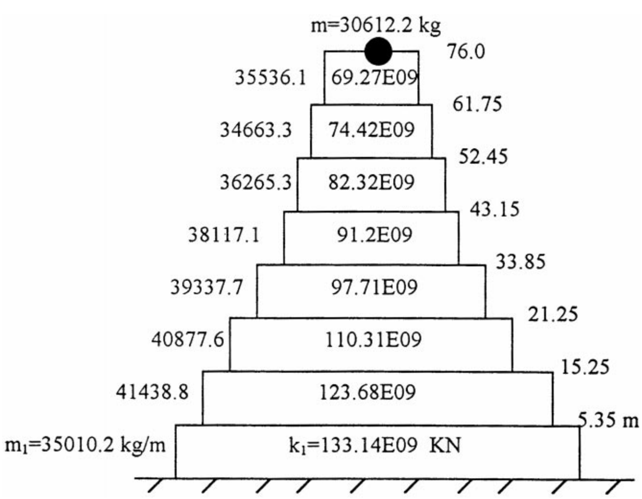

Fig. 7. The tall building is treated as a stepped bar.

in which

$$\eta_i^2=\frac{a_i H^2\omega^2}{\alpha_i},\ a_i=m_i,\ \alpha_i=K_i,\ H=76\ m \tag{80}$$

m_i and K_i are found and listed in Table 1.

Substituting Eq. (79) into Eq. (64) one yields the transfer matrices for this case as follows

$$[T_i]=\begin{bmatrix}\cos\eta_i\frac{x_{i1}}{H} & \sin\eta_i\frac{x_{i1}}{H}\\ -K_{i1}\frac{\eta_i}{H}\sin\eta_i\frac{x_{i1}}{H} & K_{i1}\frac{\eta_i}{H}\cos\eta_i\frac{x_{i1}}{H}\end{bmatrix} \tag{81}$$

$$\begin{bmatrix}\cos\eta_i\frac{x_{i0}}{H} & \sin\eta_i\frac{x_{i0}}{H}\\ -K_{i0}\frac{\eta_i}{H}\sin\eta_i\frac{x_{i0}}{H} & K_{i0}\frac{\eta_i}{H}\cos\eta_i\frac{x_{i1}}{H}\end{bmatrix}^{-1}\quad i=1,2,\ldots 7$$

$$[T_8]=\begin{bmatrix}1 & 0\\ -\omega^2 m & 1\end{bmatrix}\begin{bmatrix}\cos\eta_8\frac{x_{81}}{H} & \sin\eta_8\frac{x_{81}}{H}\\ -K_{81}\frac{\eta_8}{H}\sin\eta_8\frac{x_{81}}{H} & K_{81}\frac{\eta_8}{H}\cos\eta_8\frac{x_{81}}{H}\end{bmatrix} \tag{82}$$

$$\begin{bmatrix}\cos\eta_8\frac{x_{80}}{H} & \sin\eta_8\frac{x_{80}}{H}\\ -K_{80}\frac{\eta_8}{H}\sin\eta_8\frac{x_{80}}{H} & K_{80}\frac{\eta_8}{H}\cos\eta_8\frac{x_{80}}{H}\end{bmatrix}^{-1}$$

in which m=30 612.2 kg is the lumped mass attached to the top of the main structure of the building.

Substituting Eqs. (81) and (82) into Eq. (66) one obtains $[T]$. Setting the element T_{22} of $[T]$ equal to zero one obtains the frequency equation (i.e. Eq. (70)).

The calculated longitudinal fundamental natural frequency is 5.568 Hz. If the lumped mass attached to the top of the main structure (m=30 612.2 kg) is not considered, then, the calculated fundamental natural frequency is 5.578 Hz. The computed results of the fundamental mode shape are presented in Table 1. The longitudinal fundamental natural frequency obtained by the full-scale measurement [5] is 5.47 Hz and the measured values of the first mode shape function are also tabulated in Table 1 for comparison purposes. It is clear that the computed values in terms of the proposed procedure are in good agreement with the measured data.

If the proposed method for determining free longitudinal vibration of a one-step cantilever bar with variably distributed stiffness and mass is adopted to solve the above problem, then the step varying distributions of stiffness and mass should be changed to continuously varying distributions. It can be seen from Table 1 and Fig. 7 that the variation of the mass per unit length is comparatively small, thus, it is reasonable to assume that the mass is uniformly distributed along the height of the building [Fig. 8(a)]. The mass per unit length, $\bar{m}$, is found as: =38 014.2 kg/m.

For simplicity, the distribution of axial stiffness per unit length along the building height is described by the power function, which is given as

$$K_x=\alpha(1+\beta x)^{\gamma} \tag{83}$$

According to the real distribution of axial stiffness of this building:

at $x=0, EF_0=133.14\times10^9$ kN

$x=H, EF_H=69.27\times10^9$ kN

The parameters, α,β,γ, are determined as

$\alpha=EF_0=133.14\times10^9$ kN

$\beta=-4.825\times10^{-3}$

$\gamma=2$

The evaluated distribution of stiffness is shown in Fig. 8(c). Using the proposed formulas for determining free longitudinal vibration of a one-step cantilever bar with variably distributed stiffness and mass obtains that the fundamental natural frequency is 5.58 Hz. The calculated values of the fundamental mode shape are also listed in Table 1 (the values in square brackets). It is obvious that the difference between the results calculated by use of the step varying distributions of stiffness and mass and those obtained based on the model of a one-step cantilever bar with continuously varying stiffness and mass is so small that it can be neglected. This suggests that it is reasonable to simplify a multi-step bar with step varying distributions of stiffness and mass as a one-step bar with continuously distributed stiffness and mass for free vibration analysis when the number of step is large.

It should be noted that using the aforementioned procedure, the higher natural frequencies and corresponding mode shapes could also be determined.

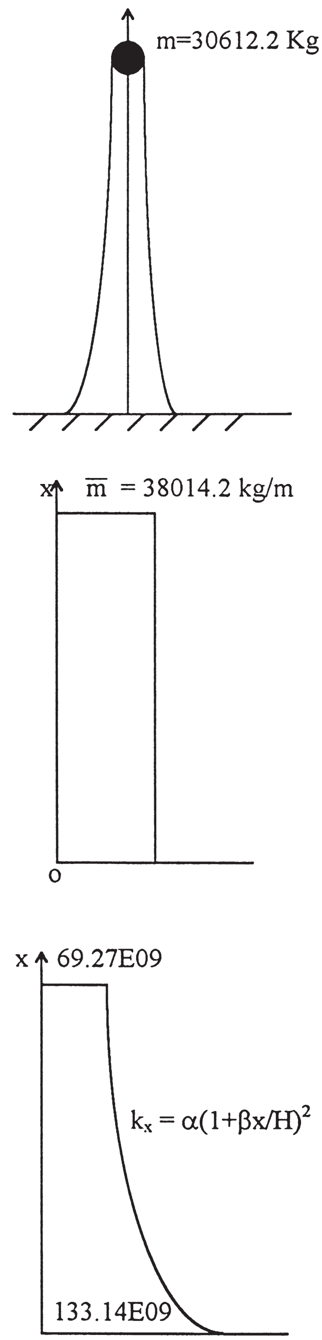

Fig. 8. (a) The tall building is simplified as a one-step bar; (b) mass distribution; (c) stiffness distribution.

5. Conclusion

The exact analytical solutions describing the longitudinal vibration of one-step bars with variably distributed stiffness and mass are derived. The obtained analytical solutions are used to establish the frequency equation of a multi-step non-uniform bar with several boundary conditions. This approach for determining structural dynamic characteristics of a multi-step non-uniform bar that combines the transfer matrix method and closed-form solutions of one-step non-uniform bars leads to a single frequency equation for any number of steps. The proposed formulae are simple and convenient for engineering applications. The numerical example showed that the calculated longitudinal fundamental natural frequency and mode shape of a 27-storey tall building are very close to the full scale measured data, suggesting that the calculation method proposed in this paper are applicable to free longitudinal vibration analysis of tall buildings. It has been demonstrated through the numerical example that the selected expressions are suitable for describing the distributions of stiffness and mass of typical tall buildings, and it is reasonable to simplify a multi-step bar with step varying distributions of stiffness and mass as a one-step bar with continuously distributed stiffness and mass for free vibration analysis when the number of steps is large.

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